Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 26, N. 2, 161-165

A Note on a Theorem of W. Gaschütz and N. Itô

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Abstract. Given a finite group G and p an odd prime number, we conclude that $\mathbf{O}^p(G) \cap G'$ is p-nilpotent when for every subgroup H of G of order p there exists a subgroup K of G such that $G = HK$ and H permutes with every subgroup of K.

Introduction

In Su [**6**] (also see Wang [**7**]), the concept of seminormality of a subgroup is introduced. Equivalently to [**6**] and [**7**] we may define: A subgroup H of a finite grupo G is said to be *semi-normal* in G if there exists a subgroup K of G such that $HK = G$ and H permutes with every subgroup of K. Clearly, every normal subgroup of G is semi-normal.

Also every subgroup of prime index is semi-normal. Our main attention we direct to finite groups in which every minimal subgroup of odd order is semi-normal.

Gaschütz and Itô ([**3**], Kap. IV, Satz 5.7) have shown that if every minimal subgroup of odd order is normal in G, then G' is p-nilpotent for each prime number $p > 2$, that is, G' has a normal Sylow-2-subgroup with nilpotent factor group.

The purpose of this note is the presentation of the following:

Theorem. *Let G be a group and p a prime number. Suppose all subgroups of order p (all cyclic subgroups of order 2 and 4 if $p = 2$) are semi-normal in G. Then $\mathbb{O}^p(G) \cap G'$ is p-nilpotent. In particular, G is p-solvable of p-length at most one. Here $\mathbb{O}^p(G)$ denotes the smallest*

Received 10 June 1994

normal subgroup of G with p-quotient.

1. Preliminary Results

We prepare the proof of the Theorem.

Lemma 1. *Let H be a semi-normal subgroup of a group G. If $H \leq L \leq G$, then H is semi-normal in L.*

For a proof see (Wang, [**7**]).

Lemma 2. *Let G be a group.*

(a) *Let p be an odd prime number. If every subgroup of order p of G is in the center of G, then G is p-nilpotent.*

(b) *If all elements of order 2 and 4 are in the center of G, then G is 2-nilpotent.*

For a proof see ([**3**], Kap. IV, Satz 5.5).

Lemma 3. *Let p be a prime number and $H \leq G$ a quasinormal p-subgroup of G. Then the group of automorphisms induced by G on H^G/H_G is a p-group.*

For a proof see (Maier-Schmid [**5**]).

If H and K are subgroup of a group G such that every subgroup of H is permutable with every subgroup of K we say that H and K are *totally permutable.*

Lemma 4. *Let $\mathcal{F}$ be a saturated formation which contains the class of supersolvable groups. Let $G = HK$ be a group such that H and K are totally permutable subgroups. If H and K lie in $\mathcal{F}$, then G is an $\mathcal{F}$-group.*

For a proof see (Maier, [**4**]).

A generalization for an arbitrary number of factors of Maier's result is given in Carocca, [**1**], also see [**2**].

Remark. Let p be a prime number. The class of all groups G such that $\mathbb{O}^p(G) \cap G'$ is p-nilpotent, is a saturated formation which contains all supersolvable groups. See ([**3**], p. 696, p. 689, Satz 6.3 and p. 716, Satz 9.1(b)).

Proof of the theorem

Theorem. *Let G be a group and p a prime number. Suppose all subgroups of order p (all cyclic subgroups of order 2 and 4 if $p = 2$) are semi-normal in G. Then $\mathbb{O}^p(G) \cap G'$ is p-nilpotent. In particular, G is p-solvable of p-length at most one.*

Proof. Let G be a group of smallest order in which the theorem is not true. Clearly the hypothesis is inherited by subgroups.

Let H denote any one of the subgroups of order p (cyclic subgroups of order 2 or 4 if $p = 2$).

For any such H we have some subgroup $K \leq G$ such that $G = HK$ and H permutes with every subgroup of K.

Case I. $K = G$ for all H.

In this case all H are quasinormal p-subgroups of G. By Lemma 3, $G/\mathbb{C}_G(H^G/H_G)$ is a p-group.

Sub-Case (i). Let $|H| = p$. If $H_G = 1$, then $\mathbb{O}^p(G) \leq \mathbb{C}_G(H^G) \leq \mathbb{C}_G(H)$. If $H \trianglelefteq G$, then $G' \leq \mathbb{C}_G(H^G) = \mathbb{C}_G(H)$.

Sub-Case (ii). Let $|H| = 4$. Let $x \in G$ be of odd order. Since H is quasinormal in G, one has $\langle H, x\rangle = H \times \langle x \rangle$, so x centralizes H.

Since $\mathbb{O}^2(G)$ is generated by the elements of odd order of G, also in this case $\mathbb{O}^2(G) \leq \mathbb{C}_G(H)$.

In any case, those of our subgroups H which are in $G' \cap \mathbb{O}^p(G)$ are central in this subgroup. So $G' \cap \mathbb{O}^p(G)$ is p-nilpotent, by Lemma 2.

Case II. For some H we have $K < G$.

Let $\mathcal{F}_p$ denote the formation of all groups G which have $G' \cap \mathbb{O}^p(G)$ p-nilpotent.

By the minimality of $|G|$, we have $K \in \mathcal{F}_p$. Also $H \in \mathcal{F}_p$.

If $|H| = p$, then K and H are totally permutable. By Lemma 4, we have $G \in \mathcal{F}_p$.

Let $|H| = 4$. If H and K are not totally permutable, then Y, the subgroup of order 2 of H is not quasinormal in G. Since Y is semi-normal in G, there exists a subgroup $L < G$ such that $G = LY$ and L and Y are totally permutable. Since Y, $L \in \mathcal{F}_p$, again $G \in \mathcal{F}_p$.

That G is p-solvable of p-length at most one follows now easily from the fact that $G/G' \cap \mathbb{O}^p(G)$ is nilpotent.

Corollary. *Let G be a group.*

(a) *If all minimal subgroups of G of odd order are semi-normal in G, then G^* has a normal Sylow-2-subgroup with nilpotent factor group. (G^* denotes the smallest normal subgroup of G with nilpotent factor group).*

(b) *If all minimal subgroups and all cyclic subgroups of order 4 of G are semi-normal, then G is a solvable group of Fitting-length at most two.*

For the proof of Corollary, we mention that $G^* = \bigcap_p \mathbb{O}^p(G) \cap G'$.

Acknowledgements

This paper was completed while the author was visiting the Departamento de Matemática of the Universidade de Brasília during the summer period January/February 1994, financially supported by the Brazilian Research Council CNPq. The author wishes to express his appreciation to Professors Helder Matos and Rudolf Maier for helpful discussions and suggestions. Also, he is very greateful for the splendid hospitality extended to him here.

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